The Probability Premium: A Graphical Representation*

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Abstract

We illustrate that Pratt's probability premium can be given a simple graphical representation allowing a direct comparison to the equivalent but more prevalent concept of risk premium under expected utility. We also show that the probability premium's graphical representation under the dual theory mimics that of the risk premium under expected utility.

Keywords: Risk Premium; Probability Premium; Expected Utility; Dual Theory; Risk Aversion.

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1 Introduction

A little more than 50 years ago, J. Pratt [11] published a very influential paper on the measurement of risk aversion in the expected utility (EU) framework.[1] He formally proposed five different definitions of risk aversion (see his Theorem 1, p. 128) and then proved that they were all mathematically equivalent.

Although they were equivalent, the five concepts didn't receive the same attention. Two of them—the risk premium and the local index of absolute risk aversion—are by far the most well-known today.

The success of the risk premium as a central concept is in part due to the fact that it can be given a simple and clear graphical interpretation in the case of zero mean binary and symmetric lotteries.[2] As a result, the concept of the risk premium nowadays appears in hundreds of textbooks and papers in economics, finance, insurance and related fields.

At the other extreme, the very intuitive concept of the probability premium (shown to be equivalent to the risk premium) has been ignored to a very wide extent. It started being used only very recently (see e.g., Jindapon [8], Liu and Meyer [9] and Liu and Neilson [10]).

The first purpose of this letter is to show that the probability premium—like the risk premium—can be given a simple graphical representation that allows a direct comparison between the two concepts inside the EU model. Besides it turns out that in Yaari's [14] dual theory (DT) of choice under risk, the graphical representation of the probability premium exactly mimics that of the risk premium under EU.

Our letter is organized as follows. The basic concepts and their notation are introduced in Section 2. Then we compare in Section 3 the graphical representation of the risk and probability premia inside the EU model. The last section, before a short conclusion, is devoted to the representation of the probability and risk premia under DT.

2 Notation and Definitions

Consider a decision maker (DM) with initial wealth x_0 and a binary lottery that pays off $\pm\varepsilon$ ($0 < \varepsilon < x_0$) with probability $\frac{1}{2}$. First, suppose this DM is an EU maximizer with an increasing and concave utility function U (i.e., $U' > 0$ and $U'' < 0$).[3] His risk premium π is defined as the solution to

$$U(x_0 - \pi) = \frac{1}{2}U(x_0 - \varepsilon) + \frac{1}{2}U(x_0 + \varepsilon). \tag{2.1}$$

As is well-known, $U'' < 0$ implies $\pi > 0$.

To define the probability premium γ the following question is asked: if the DM owns only x_0 and he is then forced to accept the binary lottery above, then by how much should the probability of the good outcome $+\varepsilon$ be increased, so that the DM's utility is preserved? Formally, γ is the solution to

$$\left(\frac{1}{2} - \gamma\right) U(x_0 - \varepsilon) + \left(\frac{1}{2} + \gamma\right) U(x_0 + \varepsilon) = U(x_0). \tag{2.2}$$

[1] As is well-known, K.J. Arrow [1, 2] independently published a similar contribution; see also the early B. de Finetti [5]. However, we refer here exclusively to Pratt [11] because our main interest is in the concept of the probability premium that appears only in Pratt [11].

[2] Surprisingly, this graphical interpretation is not found in Pratt's paper. It already appeared in Friedman and Savage's paper [6].

[3] For ease of exposition, the utility function is supposed to be twice continuously differentiable.

Turning to the dual theory (DT), we now assume that the DM distorts the probabilities by an increasing and concave function h, mapping the unit interval to itself and satisfying $h(0) = 0$ and $h(1) = 1$, while the monetary outcomes are not distorted by a utility function.

For such a DM the probability premium—that is now denoted by λ to distinguish it from the one under EU—is the solution to

$$h\left(\frac{1}{2} - \lambda\right)(x_0 - \varepsilon) + \left(1 - h\left(\frac{1}{2} - \lambda\right)\right)(x_0 + \varepsilon) = x_0. \tag{2.3}$$

Furthermore, under DT, the risk premium ρ is the solution to

$$x_0 - \rho = h\left(\frac{1}{2}\right)(x_0 - \varepsilon) + \left(1 - h\left(\frac{1}{2}\right)\right)(x_0 + \varepsilon). \tag{2.4}$$

We now proceed to the graphical representation of each of these definitions.

3 Graphical Representation of the Risk and Probability Premia in the EU Model

In Figure 1, initial wealth x_0 combined with the zero mean binary and symmetric lottery gives rise to a risk premium equal to the distance `bc`.[4]

To represent the probability premium γ we first observe that x_0 alone yields a utility level of $\mathtt{x_0d}$. Of course, if a lottery that pays off $\pm\varepsilon$ with probability $\frac{1}{2}$ is added to x_0, the DM's utility falls to $\mathtt{x_0c}$.[5] If one wants to maintain the DM's utility at its initial level, $\mathtt{x_0d}$, the probability of getting $+\varepsilon$ must be increased and become equal to the ratio $\frac{\mathtt{ae}}{\mathtt{ai}}$. Since $p = \frac{1}{2}$ corresponds to the ratio $\frac{\mathtt{ac}}{\mathtt{ai}}$ it is immediate that

$$\gamma = \frac{\mathtt{ce}}{\mathtt{ai}}.$$

Observe that the ratio $\frac{\mathtt{ce}}{\mathtt{ai}}$ is equal to the ratio $\frac{\mathtt{gh}}{\mathtt{fi}}$. Since `gh` is equal to `cd`, which is the utility premium (see footnote 5), we can also write:

$$\gamma = \frac{\text{utility premium}}{U(x_0 + \varepsilon) - U(x_0 - \varepsilon)}. \tag{3.1}$$

Because U is defined up to a positive linear transformation, the denominator in (3.1) can be made equal to unity without loss of generality, and one then obtains that the probability premium coincides with the utility premium, `cd`.

Note finally that if U becomes more concave (while still running through the points a and i) both π and γ increase as claimed by Pratt.

[4] Throughout, distances are written in teletype font.

[5] The distance `cd` is termed the utility premium by Friedman and Savage [6]; see also e.g., the more recent Eeckhoudt and Schlesinger [4], Huang and Stapleton [7] and the references therein.

4 The Probability and Risk Premia in the Dual Model

Under DT, the probability premium, denoted by λ, is defined in (2.3), which can also be written as:

$$-2h\left(\frac{1}{2}-\lambda\right)\varepsilon+\varepsilon=0 \qquad \text{or} \qquad h\left(\frac{1}{2}-\lambda\right)=\frac{1}{2}. \tag{4.1}$$

Equation (4.1) lends itself to an interesting graphical representation. In Figure 2, the function h is concave which corresponds to (strong) risk aversion.[6] From (4.1) we have to find a probability level $\frac{1}{2}-\lambda$ such that $h\left(\frac{1}{2}-\lambda\right)=\frac{1}{2}$ and it is easily seen that the probability premium $\lambda = \texttt{bc}$. The graphical representation of the probability premium in the dual model exactly mimics that of the risk premium under EU.

To represent the risk premium ρ under DT one starts with (2.4), which can be re-written as:

$$\rho = 2\left(h\left(\frac{1}{2}\right)-\frac{1}{2}\right)\varepsilon. \tag{4.2}$$

In (4.2) ρ is proportional to ε, a measure of the quantity of risk (standard deviation) for this binary lottery. To obtain ρ, the quantity of risk ε is multiplied by two times a coefficient of risk aversion represented by `cd` in Figure 2.

Finally, note that if the probability distortion function h becomes more concave, both λ and ρ increase.

5 Conclusion

Among the five equivalent definitions of the degree of risk aversion in the EU model, the most popular one is that of the risk premium. One of the reasons for its success is its well-known simple graphical representation in the case of zero mean binary and symmetric lotteries.

In this letter, we have shown that the definition of risk aversion based on the probability premium can also be given a simple and parallel graphical representation. Since the notion of the probability premium, which was practically ignored for 50 years, starts receiving attention, this letter may turn out to be useful for its development.

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[6] See Chew, Karni and Safra [3] and Roëll [13] for further details on definitions and implications of risk aversion under DT and under Quiggin's [12] generalization.

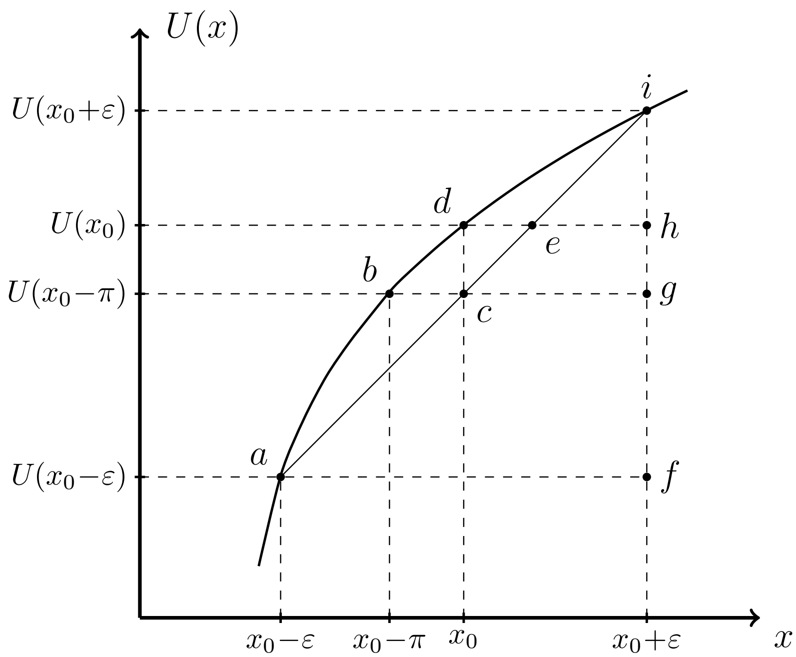

Figure 1: Graphical Representation of the Risk and Probability Premia under the EU Model

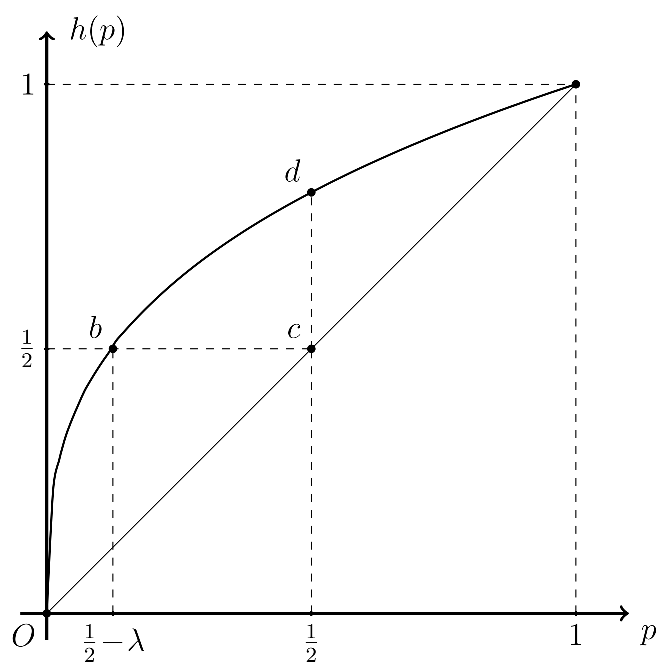

Figure 2: Graphical Representation of the Probability and Risk Premia under the DT Model

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