Geometrically Convex Return Risk Measures and Orlicz Premia*

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June 19, 2024

Abstract

This paper studies return risk measures (i.e., normalized, monotone and positively homogeneous functionals) and geometric convexity (i.e., convexity with respect to the geometric mean) that appears naturally in geometric risk evaluation. We first provide general and law-invariant dual representation results for geometrically convex and convex return risk measures. Next, we restrict attention to Orlicz premia (i.e., Luxemburg norms) of which we introduce a more general definition than conventionally, including also e.g., geometric means, i.e., logarithmic certainty equivalents, and expectiles as particular cases. We establish new characterization results for Orlicz premia. We demonstrate that translation invariant Orlicz premia take the form of L^p-quantiles, reducing to expectiles when additionally convexity is required. We also provide a novel axiomatization of Orlicz premia as the only elicitable geometrically convex or convex return risk measures.

Keywords Return risk measures, geometric convexity, Orlicz premia, L^p-quantiles and expectiles, elicitability.

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*We are very grateful to Marco Frittelli, Emanuela Rosazza Gianin and to seminar and conference participants at the University of Vienna, the University of Amsterdam, the University of Ulm, Heriott-Watt University, the Amsterdam-Leuven-London (ALL) workshop in Amsterdam and the 11th General AMAMEF conference in Bielefeld for their comments and suggestions. This research was funded in part by the Netherlands Organization for Scientific Research under an NWO-Vici grant 2020–2027 (Aygün and Laeven).

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1 Introduction

Recently, [7] introduced the class of return risk measures, consisting of normalized, monotone and positively homogeneous functionals. Return risk measures provide relative (or geometric) assessments of risk. They evaluate how much additional riskless log-return makes a financial position acceptable—whence their name. They constitute the relative counterparts of the class of monetary risk measures ([23, 18]), reminiscent of how relative risk aversion relates to absolute risk aversion. Their dynamic extensions, dynamic return risk measures, have been studied in [8].[1]

This paper makes three main contributions. We start by providing dual representation results for geometrically convex and convex return risk measures and clarify their precise relationship involving the relative entropy. In full generality, the dual representation takes the form of a supremum of discounted logarithmic certainty equivalents, where the discount factor can be interpreted as an index of model plausibility under ambiguity. We show that convex return risk measures occur as a special case in the richer class of geometrically convex return risk measures. Furthermore, we introduce and analyze the class of optimized return (OR, for short) risk measures. We also analyze corresponding law-invariant representations and related continuity properties.

Second, we provide a new definition of Orlicz premia by extending the class of loss functions that generates them. Orlicz premia, also known as Luxemburg norms, and the intimate links between risk measures and Orlicz space theory, have been extensively studied in the actuarial and financial mathematics literature (see e.g., [27, 11, 15, 16, 18, 35, 7, 8] and the references therein). Our definition is (substantially) more general than the conventional one ([27]) and encompasses many interesting families of return risk measures that are traditionally excluded. Examples include geometric means, i.e., logarithmic certainty equivalents, expectiles and L^p-quantiles. We derive a variety of properties of our Orlicz premia.

Third, we establish new characterization results for our Orlicz premia. First, we demonstrate that the class of L^p-quantiles arises as the class of Orlicz premia that are translation invariant. We also show that, upon additionally requiring convexity (hence, yielding 'coherent' risk measures), the L^p-quantiles reduce to the class of expectiles. These results stand in sharp contrast to classical results about conventional Orlicz premia (based on differentiable Young functions, see [27, 25]) that assert that convex Orlicz premia are translation invariant if and only if they 'collapse to the mean'. We also provide characterizations and dual representations of geometrically convex and convex Orlicz premia, exploiting our earlier dual representation results. Furthermore, we prove that Orlicz premia naturally arise as the only return risk measures that are elicitable. An expanding and increasingly sophisticated literature has studied elicitability properties of risk measures (e.g., [44, 40, 24, 47, 19, 22]). It has been shown in [40] that

[1]Return risk measures that allow for probability distortion were recently analyzed in [46], star-shaped generalizations were recently studied in [30, 31, 32, 33], whereas applications of return risk measures to capital allocation can be found in [36] and [12]; see also [4].

an elicitable risk measure must satisfy the convex level sets (CxLS) property. We establish that a law-invariant geometrically convex or convex return risk measure with the CxLS property is necessarily an Orlicz premium.

The rest of this paper is organized as follows. In Section 2, we recall the general properties of return risk measures and derive some useful geometric convexity and continuity properties. In Section 3, we provide dual representation results for geometrically convex and convex return risk measures, explicate their connection and analyze optimized return risk measures. Section 4 derives corresponding law-invariant representations and related continuity properties. In Section 5, we introduce our generalized definition of Orlicz premia and establish our characterization results for Orlicz premia. All proofs are in the Appendix.

2 Return risk measures

Let $(\Omega, \mathcal{F}, P)$ be a nonatomic probability space. In the present paper, random variables $X\colon \Omega \to \mathbb{R}$ represent financial losses. We consider finite-valued risk measures defined on $L^\infty(\Omega, \mathcal{F}, P)$ or on its subsets $L^\infty_+(\Omega, \mathcal{F}, P) := \{X \in L^\infty \mid X \geqslant 0\ P\text{-a.s.}\}$ and $L^\infty_{++}(\Omega, \mathcal{F}, P) := \{X \in L^\infty_+ \mid X \geqslant c > 0\ P\text{-a.s.}\}$. Equalities and inequalities between random variables are meant to hold P-a.s. without further mentioning. We start by recalling basic definitions.

Definition 1 *A risk measure $\rho\colon L^\infty(\Omega, \mathcal{F}, P) \to \mathbb{R}$ is:*

a) translation invariant if $\rho(X + h) = \rho(X) + h$, $\forall h \in \mathbb{R}$, $\forall X \in L^\infty$

b) monotone if $X \leqslant Y \Rightarrow \rho(X) \leqslant \rho(Y)$

c) monetary if ρ is translation invariant, monotone and satisfies $\rho(0) = 0$

d) positively homogeneous if $\rho(\lambda X) = \lambda\rho(X)$, $\forall \lambda \geqslant 0$, $\forall X \in L^\infty$

e) convex if $\rho(\alpha X + (1-\alpha)Y) \leqslant \alpha\rho(X) + (1-\alpha)\rho(Y)$, $\forall X, Y \in L^\infty$, $\forall \alpha \in (0, 1)$

f) coherent if it is monetary, convex and positively homogeneous

g) law invariant if $X \overset{d}{=} Y \Rightarrow \rho(X) = \rho(Y)$, where $X \overset{d}{=} Y$ if X and Y have the same distribution.

Furthermore, we say that:

h) ρ has the Fatou property if $X_n \overset{P}{\to} X$, $\|X_n\|_\infty \leqslant k \implies \rho(X) \leqslant \liminf_{n\to+\infty} \rho(X_n)$

i) ρ has the Lebesgue property if $X_n \overset{P}{\to} X$, $\|X_n\|_\infty \leqslant k \implies \rho(X_n) \to \rho(X)$.

A law-invariant risk measure on $L^\infty(\Omega, \mathcal{F}, P)$ induces a functional on $\mathcal{M}_{1,c}(\mathbb{R})$, the set of probability measures with compact support in $\mathbb{R}$, by means of $\rho(F) := \rho(X)$ if X has distribution F, and each $\mu \in \mathcal{M}_{1,c}(\mathbb{R})$ is identified with its distribution $F(x) := \mu(-\infty, x]$. We also recall from [7] the notions of return risk measure and multiplicative acceptance set.

Definition 2 *A* return risk measure $\tilde{\rho}\colon L^\infty_+ \to [0,+\infty)$ *is a positively homogeneous and monotone risk measure satisfying* $\tilde{\rho}(1) = 1$*. Its multiplicative acceptance set is*

$$B_{\tilde{\rho}} = \{X \in L^\infty_+ \mid \tilde{\rho}(X) \leqslant 1\}.$$

For return risk measures the notion of geometric convexity, also known as convexity with respect to the geometric mean, multiplicative convexity or GG-convexity in the case of functions on the positive real line (see e.g., [38, 39]), will be central in the paper.

Definition 3 *A risk measure* $\tilde{\rho}\colon L^\infty_+ \to [0,+\infty)$ *is* geometrically convex *if for each* $X, Y \in L^\infty_+$ *and* $\alpha \in (0,1)$ *it holds that*

$$\tilde{\rho}(X^\alpha Y^{1-\alpha}) \leqslant \tilde{\rho}^\alpha(X)\tilde{\rho}^{1-\alpha}(Y).$$

As a consequence of positive homogeneity and of the AM-GM inequality, a convex return risk measure is also geometrically convex, as the next lemma shows.

Lemma 4 *If* $\tilde{\rho}\colon L^\infty_+ \to [0,+\infty)$ *is a convex return risk measure, then it is geometrically convex.*

The class of geometrically convex return risk measures is thus larger than the class of convex return risk measures. It is indeed strictly larger, and the simplest example of a geometrically convex risk measure that is not convex is the geometric mean, sometimes also called logarithmic certainty equivalent, defined by $\tilde{\rho}(X) = \exp \mathbb{E}[\log X]$. As we will see, the geometric mean will play a fundamental role in the theory of geometrically convex risk measures.

A one-to-one correspondence between return risk measures and monetary risk measures has been outlined in [7] as follows: given a monetary risk measure $\rho\colon L^\infty \to \mathbb{R}$, the associated return risk measure $\tilde{\rho}\colon L^\infty_{++} \to (0,+\infty)$ is given by

$$\tilde{\rho}(X) := \exp\left(\rho\left(\log(X)\right)\right), \tag{1}$$

and, *vice versa*, given a return risk measure $\tilde{\rho}\colon L^\infty_{++} \to (0,+\infty)$, the associated monetary risk measure $\rho\colon L^\infty \to \mathbb{R}$ is

$$\rho(Y) := \log\left(\tilde{\rho}\left(\exp(Y)\right)\right). \tag{2}$$

The main properties of this correspondence are recalled in the following lemma from [7].

Lemma 5 *Let* $\rho\colon L^\infty \to \mathbb{R}$ *and* $\tilde{\rho}\colon L^\infty_{++} \to (0,+\infty)$ *be related by* (1) *and* (2)*. Then:*

a) $\rho(0) = 0 \iff \tilde{\rho}(1) = 1$

b) ρ *is translation invariant* $\iff$ $\tilde{\rho}$ *is positively homogeneous*

c) if $A_\rho := \{X \in L^\infty \mid \rho(X) \leqslant 0\}$ *is the acceptance set of* ρ*, then* $B_{\tilde{\rho}} = \exp(A_\rho)$

d) ρ *is monotone* $\iff$ $\tilde{\rho}$ *is monotone*

e) ρ *is convex* $\iff$ $\tilde{\rho}$ *is geometrically convex*

f) ρ *is law invariant* $\iff$ $\tilde{\rho}$ *is law invariant*

g) *if* ρ *is law invariant, then, for* $F \in \mathcal{M}_{1,c}(0, +\infty)$,

$$\tilde{\rho}(F) = \exp\left(\rho\left(F(e^t)\right)\right). \tag{3}$$

We refer to [7, 8] for many examples and applications of this correspondence.

Our first aim is to derive a dual representation for geometrically convex return risk measures, which will be given in Theorem 8 of Section 3. It will turn out that for return risk measures the definitions of Fatou and Lebesgue properties have to be slightly modified.

Definition 6 *A return risk measure* $\tilde{\rho}$ *satisfies*

a) *the* lower-bounded Fatou property *if*

$$X_n \xrightarrow{P} X,\ \|X_n\|_\infty \leqslant k, X_n \geqslant c > 0 \implies \tilde{\rho}(X) \leqslant \liminf_{n\to+\infty} \tilde{\rho}(X_n),$$

b) *the* lower-bounded Lebesgue property *if*

$$X_n \xrightarrow{P} X,\ \|X_n\|_\infty \leqslant k, X_n \geqslant c > 0 \implies \tilde{\rho}(X_n) \to \tilde{\rho}(X).$$

Clearly, the lower-bounded Lebesgue property implies the lower-bounded Fatou property. Both properties are weaker than the usual ones, asserting lower semicontinuity and continuity, respectively, under more restrictive assumptions. The lower-bounded Fatou and Lebesgue properties of $\tilde{\rho}$ are equivalent to the usual Fatou and Lebesgue properties of the associated ρ.

Lemma 7 *Let* $\rho\colon L^\infty \to \mathbb{R}$ *and* $\tilde{\rho}\colon L^\infty_{++} \to (0, +\infty)$ *be as in* (1) *and* (2)*. Then:*

(i) $\tilde{\rho}$ *has the lower-bounded Fatou property if and only if* ρ *has the Fatou property*

(ii) $\tilde{\rho}$ *has the lower-bounded Lebesgue property if and only if* ρ *has the Lebesgue property.*

3 Dual representations

In this section, we give a dual representation of geometrically convex return risk measures and compare it with a similar formula for convex return risk measures derived in [34]. We introduce preliminarily a few additional notations. If P, Q are probability measures on $(\Omega, \mathcal{F})$, we say that Q is absolutely continuous with respect to P and we write $Q \ll P$ if, $\forall A \in \mathcal{F}$, $P(A) = 0 \Rightarrow Q(A) = 0$. We set $\mathbf{P} := \{Q \text{ on } (\Omega, \mathcal{F}) \text{ such that } Q \ll P\}$. Recall also that $B_{\tilde{\rho}}$ is the multiplicative acceptance set of $\tilde{\rho}$ introduced in Definition 2.

Theorem 8 *Let* $\tilde{\rho}\colon L^{\infty}_{++} \to (0, +\infty)$ *be a geometrically convex return risk measure satisfying the lower-bounded Fatou property. Then,*

$$\tilde{\rho}(X) = \sup_{Q \in \mathbf{P}} \{\tilde{\alpha}(Q) \exp\left(\mathbb{E}_Q[\log X]\right)\}, \tag{4}$$

where $\tilde{\alpha}\colon \mathbf{P} \to [0, 1]$ *is given by*

$$\tilde{\alpha}(Q) = \left[\sup_{X \in B_{\tilde{\rho}}} \exp(\mathbb{E}_Q[\log X]) \right]^{-1}. \tag{5}$$

If $\tilde{\rho}$ *satisfies the lower-bounded Lebesgue property, then the supremum in* (4) *is attained.*

This theorem shows that any geometrically convex return risk measure can be expressed as the supremum of multiplicatively weighted (or discounted) geometric means, computed with respect to different probability measures $Q \in \mathbf{P}$. The less plausible the probabilistic model $Q \in \mathbf{P}$, the lower is the corresponding weight $\alpha(Q)$. As we will discuss in Example 25 of Section 5, the geometric mean $\exp\left(\mathbb{E}[\log X]\right)$ is an example of a generalized notion of Orlicz premium corresponding to the unbounded and non-convex Orlicz function $\Phi(x) = 1 + \log(x)$.

Theorem 8 thus shows that every geometrically convex return risk measure satisfying the lower-bounded Fatou property can be expressed as the supremum of a suitable family of multiplicatively weighted Orlicz premia, underlining the role of Orlicz premia as the basic building blocks of general geometrically convex return risk measures. In the related working paper [3], we show that dual representations similar to (4) and (5) hold for general geometrically convex functions, and are instances of a generally defined geometrically convex transform.

In the next proposition we give a similar multiplicatively weighted dual representation of convex return risk measures that can be easily derived from Proposition 4.3 of [34].

Proposition 9 *Let* $\tilde{\rho}\colon L^{\infty}_{+} \to [0, +\infty)$ *be a convex return risk measure satisfying the Fatou property. Then,*

$$\tilde{\rho}(X) = \sup_{Q \in \mathbf{P}} \{\tilde{\beta}(Q)\, \mathbb{E}_Q[X]\}, \tag{6}$$

where

$$\tilde{\beta}(Q) = \left[\sup_{X \in B_{\tilde{\rho}}} \mathbb{E}_Q[X] \right]^{-1}.$$

If $\tilde{\rho}$ *satisfies the Lebesgue property, then the supremum in* (6) *is attained.*

It is very interesting to compare the dual representations given in equations (4) and (6). Since, as we have seen in Lemma 4, a convex return risk measure is also geometrically convex, the dual representation given in (4) is implied by the one given in (6), and the link between $\tilde{\alpha}$ and $\tilde{\beta}$ is given in Proposition 11 below. Recall first the following ([17]).

Definition 10 *If R, Q are probability measures on (Ω, P), then the relative entropy of R with respect to Q is given by*

$$\mathrm{H}(R,Q) := \begin{cases} \mathbb{E}_Q\left[\frac{\mathrm{d}R}{\mathrm{d}Q}\log\frac{\mathrm{d}R}{\mathrm{d}Q}\right] & \text{if } R \ll Q \\ +\infty & \text{otherwise.} \end{cases}$$

Proposition 11 *Let $\tilde{\rho}\colon L^\infty_{++} \to (0,+\infty)$ be a convex return risk measure with a dual representation given as in (6). Then $\tilde{\rho}$ admits a dual representation as a GG-convex risk measure with*

$$\tilde{\alpha}(R) = \sup_{Q \ll P}\left\{\frac{\tilde{\beta}(Q)}{\exp(\mathrm{H}(R,Q))}\right\}.$$

We end the section by describing how return risk measures can be optimized, following a similar construction as for optimized certainty equivalents introduced in [9, 10].

Definition 12 *Let $\tilde{\rho}\colon L^\infty_{+} \to [0,+\infty)$ be a return risk measure. The corresponding optimized return risk measure (henceforth, OR risk measure) $\rho : L^\infty \to \mathbb{R}$ is defined by*

$$\rho(X) = \inf_{x\in\mathbb{R}}\left\{x + \tilde{\rho}\left((X-x)^+\right)\right\}. \tag{7}$$

The special case in which $\tilde{\rho}$ is an Orlicz premium and the corresponding ρ is a Haezendonck-Goovaerts (HG) risk measure has been considered in [26, 6]. The class of OR risk measures is rich and encompasses as special cases the Rockafellar-Uryasev [42] construction of Average-Value-at-Risk as well as its generalizations given by the HG and robust HG risk measures ([7]). As in the case of Orlicz premia, optimized return risk measures are translation invariant and inherit the convexity property.

Lemma 13 *An OR risk measure satisfies the following properties:*

a) monotonicity

b) positive homogeneity

c) translation invariance

d) if $\tilde{\rho}$ is convex, then ρ is convex.

In the convex case, as outlined in Section 4 of [6] for the special case of Orlicz premia, optimized risk measures can be expressed as inf-convolutions, from which their dual representation can easily be derived. We just recall the definition and the basic idea in the following lemma and omit further details for brevity.

Definition 14 *The inf-convolution $(f \,\square\, g)$ of two functionals $f\colon L^\infty \to \overline{\mathbb{R}}$ and $g\colon L^\infty \to \overline{\mathbb{R}}$ is defined as follows:*

$$(f \,\square\, g)(X) = \inf_{Y\in L^\infty}\left\{f(X-Y) + g(Y)\right\}.$$

Lemma 15 *An OR risk measure ρ can be written as*

$$\rho(X) = (f \,\square\, g)(X),$$

where $f(X) = \tilde{\rho}(X^+)$ and

$$g(Y) = \begin{cases} x & \text{if } Y = x, \\ +\infty & \text{otherwise,} \end{cases}$$

when the corresponding return risk measure $\tilde{\rho}$ is convex.

On the contrary, the property of geometric convexity is not inherited by the OR risk measure associated to ρ, as the following simple example shows.

Example 16 *Let*

$$\tilde{\rho}(X) = \begin{cases} \exp(\mathbb{E}[\log X]) & \text{if } P(X=0) = 0 \\ 0 & \text{if } P(X=0) > 0 \end{cases}.$$

Then, $\tilde{\rho}$ is geometrically convex and equation (7) in Definition 12 gives

$$\rho(X) = \inf_{x \in \mathbb{R}} \left\{ x + \exp\left(\mathbb{E}[\log(X - x)^+]\right) \right\}.$$

Letting $X = \frac{1}{2} \cdot 1_A + 2 \cdot 1_{A^c}$ with $P(A) = 1/2$, it is not difficult to verify that $\rho(X) = 1/2$. Setting $Y = 1/X$, it follows that $X \overset{d}{=} Y$ and $XY = 1$, so we have

$$1 = \rho(X^{1/2}Y^{1/2}) > [\rho(X)]^{1/2} \cdot [\rho(Y)]^{1/2} = 1/2,$$

which shows that ρ is not geometrically convex.

4 The law-invariant case

We now focus on law-invariant geometrically convex return risk measures. The first result is a Kusuoka-like representation given in Theorem 18. Recall first the definition of Average Value-at-Risk.

Definition 17 *Let $X \in L^1(\Omega, \mathcal{F}, P)$. For $\lambda \in [0,1)$, the Average Value-at-Risk of X at level λ is given by*

$$AV@R_\lambda(X) = \frac{1}{1-\lambda} \int_\lambda^1 q_\alpha(X)\, \mathrm{d}\alpha,$$

where

$$q_\alpha(X) = \inf\{x \in \mathbb{R} \mid F(x) \geqslant \alpha\}.$$

For $\lambda = 1$, we set by definition $AV@R_1(X) = \operatorname{ess\,sup}(X)$.

Denote by $\mathcal{M}_1([0,1])$ the set of probability measures with support in $[0,1]$.

Theorem 18 *Let $\tilde{\rho}\colon L^{\infty}_{++} \to (0,+\infty)$ be a law-invariant geometrically convex return risk measure. Then there exists $\tilde{\beta}\colon \mathcal{M}_1([0,1]) \to [0,1]$ such that*

$$\tilde{\rho}(X) = \sup_{\mu \in \mathcal{M}_1([0,1])} \left\{ \tilde{\beta}(\mu) \exp\left(\left(\int_{[0,1]} AV@R_\lambda(\log X)\mu(\mathrm{d}\lambda) \right) \right) \right\}.$$

If $\tilde{\rho}$ has the lower-bounded Lebesgue property, then $\mu(1) > 0 \Rightarrow \tilde{\beta}(\mu) = 0$.

Next, we discuss continuity properties of law-invariant geometrically convex risk measures, seen as functionals on $\mathcal{M}_{1,c}(\mathbb{R})$. Since a law-invariant, monetary and convex risk measure on L^∞ automatically satisfies the Fatou property (see e.g., [28] and [14]), it follows from Lemma 7 that a law-invariant geometrically convex return risk measure automatically has the lower-bounded Fatou property. As a consequence, we have the following *partial* mixture continuity result, where δ_x denotes as usual a probability measure supported at x.

Proposition 19 *Let $\tilde{\rho}\colon \mathcal{M}_{1,c}(0,+\infty) \to (0,+\infty)$ be a law-invariant geometrically convex return risk measure. Let $0 < x < y$. Then the mapping*

$$\lambda \mapsto \tilde{\rho}(\lambda\delta_x + (1-\lambda)\delta_y)$$

is continuous at each $\lambda \in (0,1)$ and for $\lambda \to 0^+$.

Mixture-continuity may fail for $\lambda \to 1^-$, an example being $\tilde{\rho}(X) = \operatorname{ess\,sup}(X)$, since

$$\tilde{\rho}(\lambda\delta_x + (1-\lambda)\delta_y) = \begin{cases} y & \text{if } 0 \leqslant \lambda < 1 \\ x & \text{if } \lambda = 1. \end{cases}$$

However, we show below that, if $\tilde{\rho}$ satisfies the lower-bounded Lebesgue property, then mixture-continuity holds also for $\lambda \to 1^-$. Furthermore, $\tilde{\rho}$ then satisfies the property of ψ-weak continuity defined below. We refer to [23] and to [29] and [19] and the references therein for more on ψ-weak continuity and its relevance for financial risk measures.

Definition 20 *Let $\psi\colon \mathbb{R} \to [1,+\infty)$ be continuous. The ψ-weak topology on $\mathcal{M}_{1,c}(\mathbb{R})$ is the weakest topology that makes all mappings $F \mapsto \int f\,\mathrm{d}F$ continuous, for each continuous f satisfying $|f| \leqslant c\psi$, with $c > 0$. It holds that*

$$F_n \xrightarrow{\psi} F \text{ if } F_n \xrightarrow{weakly} F \text{ and } \int \psi\,\mathrm{d}F_n \to \int \psi\,\mathrm{d}F.$$

A functional $\rho\colon \mathcal{M}_{1,c} \to \mathbb{R}$ is ψ-weakly continuous if

$$F_n \xrightarrow{\psi} F \implies \rho(F_n) \to \rho(F).$$

As for the case of convex risk measures studied in [19] (see also [18]), the validity of the lower-bounded Lebesgue property is linked to a weak compactness property of the upper level sets of the multiplicative weighting function $\tilde{\alpha}(Q)$ given in equation (4) of Theorem 8.

Definition 21 *A geometrically convex return risk measure $\tilde{\rho}$ with dual representation (4) has the $\widetilde{WC}$ property if for each $m > 0$ the upper level sets $\{Q \in \mathbf{P} \mid \tilde{\alpha}(Q) \geqslant m\}$ are compact in the $\sigma(L^1, L^\infty)$-topology.*

Proposition 22 *Let $\tilde{\rho} : \mathcal{M}_{1,c}(\mathbb{R}) \to (0, +\infty)$ be a law-invariant geometrically convex return risk measure. The following statements are equivalent:*

a) $\tilde{\rho}$ has the lower-bounded Lebesgue property

b) $\tilde{\rho}$ has the $\widetilde{WC}$ property

c) $\tilde{\rho}$ is $\tilde{\psi}$-weakly continuous for some $\tilde{\psi}\colon (0, +\infty) \to \mathbb{R}$

d) $\tilde{\rho}$ is (fully) mixture-continuous.

We already mentioned that $\tilde{\rho}(X) = \operatorname{ess\,sup}(X)$ is not fully mixture-continuous. We consider the mixture-continuity properties of two other examples.

Example 23 *If $\tilde{\rho}(x) = \exp(\mathbb{E}[\log X])$, then*

$$\tilde{\alpha}(Q) = \begin{cases} 1 & \text{if } Q = P \\ 0 & \text{if } Q \neq P \end{cases}$$

so clearly condition $\widetilde{WC}$ is satisfied, and indeed also mixture-continuity is trivially satisfied.

Example 24 *If, more generally,*

$$\tilde{\alpha}(Q) = \begin{cases} 1 & \text{if } Q \in \mathcal{M} \\ 0 & \text{if } Q \notin \mathcal{M} \end{cases}$$

then the condition $\widetilde{WC}$ holds if and only if the set $\mathcal{M}$ is $\sigma(L^1, L^\infty)$-compact. Notice that here

$$\tilde{\rho}(X) = \exp(\rho(\log X)),$$

where $\rho(X) = \sup_{Q \in \mathcal{M}} \mathbb{E}_Q[X]$ is a coherent risk measure with the Lebesgue property.

5 Orlicz premia

An important family of return risk measures displaying interesting convexity and geometric convexity properties is constituted by Orlicz premia (i.e., Luxemburg norms), introduced in the actuarial literature in [27]. They are defined by

$$H_\Phi(X) = \inf\{k > 0 | \, \mathbb{E}\left[\Phi(X/k)\right] \leqslant 1\},$$

where the so-called (normalized) Young function $\Phi\colon [0, +\infty) \to [0, +\infty)$ is strictly increasing on $(\Phi > 0)$ and convex and satisfies $\Phi(0) = 0$ and $\Phi(1) = 1$. We refer to [41, 21] for the basic properties of Luxemburg norms under these, and related, assumptions.

5.1 A generalized definition and its properties

We start by noticing that the notion of a Young function can be substantially generalized to include among Orlicz premia the following remarkable examples.

Example 25 (Geometric mean) *Let* $\Phi(x) = 1 + \log(x)$*. Then,*

$$H_\Phi(X) = \inf\left\{k > 0 \;\middle|\; \mathbb{E}\left[1 + \log\left(\frac{X}{k}\right)\right] \leqslant 1\right\} = \exp\left(\mathbb{E}\left[\log X\right]\right).$$

In this example, Φ *is concave and* $\Phi(0) = -\infty$*.*

Example 26 (Quantiles) *Let*

$$\Phi_q(x) = \begin{cases} q & \text{if } 0 \leqslant x \leqslant 1 \\ 1+q & \text{if } x > 1 \end{cases}$$

with $0 < q < 1$*. Then the Orlicz premium* $H_{\Phi_q}(X)$ *coincides with the left* q*-quantile of* X*. In this example,* Φ_q *is non-decreasing, not convex and satisfies* $\Phi_q(0) = q > 0$ *and* $\Phi_q(1) = q < 1$*.*

Example 27 (Expectiles) *Let* $\Phi_q(x) = 1 + q(x-1)_+ - (1-q)(x-1)_-$ *with* $0 < q < 1$*. Then,*

$$\mathbb{E}\left[\Phi\left(\frac{X}{k}\right)\right] \leqslant 1 \iff q\,\mathbb{E}\left[\left(\frac{X}{k} - 1\right)_+\right] \leqslant (1-q)\,\mathbb{E}\left[\left(\frac{X}{k} - 1\right)_-\right],$$

so $H_{\Phi_q}(X) = e_q(X)$*, which is the* q*-expectile of* X *introduced in [37] given by*

$$e_q(X) = \inf\{k \in \mathbb{R} \mid q\,\mathbb{E}[(X-k)_+] \leqslant (1-q)\,\mathbb{E}[(X-k)_-].$$

Here, $\Phi_q(0) = q > 0$ *and* Φ_q *is convex if* $1/2 \leqslant q < 1$ *and concave if* $0 < q \leqslant 1/2$*. Notice also that* Φ_q *is not differentiable in* $x = 1$ *if* $q \neq 1/2$*.*

Example 28 (L^p-quantiles) *More generally, let* $\Phi_{q,p}(x) = 1 + q(x-1)_+^p - (1-q)(x-1)_-^p$ *with* $p > 0$ *and* $0 < q < 1$*. Then,* $H_{\Phi_{q,p}}$ *is the solution of the equation*

$$q\,\mathbb{E}\left[\left(\frac{X}{H} - 1\right)_+^p\right] = (1-q)\,\mathbb{E}\left[\left(\frac{X}{H} - 1\right)_-^p\right],$$

so it is a L^{p+1}*-quantile in the sense of [13].*

These examples suggest to extend the notion of Orlicz premium by considering a more general class of loss functions, which we call Orlicz functions in the definition below to distinguish them from classical Young functions.

Definition 29 *An Orlicz function* $\Phi\colon [0, +\infty) \to \mathbb{R} \cup \{\pm\infty\}$ *satisfies:*

a) $\Phi(x) > -\infty$ *if* $x > 0$*,* $\Phi(x) \leqslant 1$ *if* $x \leqslant 1$*,* $\Phi(x) > 1$ *if* $x > 1$

b) Φ *is nondecreasing*

c) Φ *is left-continuous.*

For $X \in L^\infty_+$*, the Orlicz premium is defined by*

$$H_\Phi(X) = \inf\{k > 0 \mid \mathbb{E}\left[\Phi(X/k)\right] \leqslant 1\}.$$

If $P(X = 0) > 0$ *and* $\Phi(0) = -\infty$*, we set by definition* $H_\Phi(X) = 0$*.*

A remarkable special case of Definition 29 is the essential supremum, which can be obtained with many different Orlicz functions.

Example 30 *Let* $X \in L^\infty_{++}$ *and*

$$a)\ \Phi_1(x) = \begin{cases} 1 & \textit{if } 0 < x \leqslant 1 \\ f(x) & \textit{if } x > 1 \end{cases}, \quad b)\ \Phi_2(x) = \begin{cases} g(x) & \textit{if } 0 < x \leqslant 1 \\ +\infty & \textit{if } x > 1 \end{cases},$$

where $f\colon (1, +\infty) \to (1, +\infty)$ *and* $g\colon (0, 1] \to (-\infty, 1]$ *are chosen in order to satisfy Definition 29. Then,* $H_{\Phi_1}(X) = H_{\Phi_2}(X) = \operatorname{ess\,sup}(X)$*.*

The properties required to the Orlicz function Φ in Definition 29 are necessary to preserve the most relevant properties of Orlicz premia, with the exception of convexity. One important difference with respect to the usual notion of a Young function is that, if Φ is not convex, then the set $\{X \mid \mathbb{E}[\Phi(X/k)] \leqslant 1\}$ is not necessarily convex any more.

Proposition 31 *Let* Φ *and* $H_\Phi(X)$ *be as in Definition 29. Then,*

a) H_Φ *is monotone, positively homogeneous and satisfies* $H_\Phi(1) = 1$

b) If $u := \sup\{x \mid \Phi(x) < +\infty\}$*, then*

$$\frac{\operatorname{ess\,sup} X}{u} \leqslant H_\Phi(X) \leqslant \operatorname{ess\,sup} X$$

c) If $H_\Phi(X) > 0$*, then* $H_\Phi(X) = \min\{k > 0 \mid \mathbb{E}[\Phi(X/k)] \leqslant 1\}$

d) It holds that $H_\Phi(X) \leqslant 1 \iff \mathbb{E}[\Phi(X)] \leqslant 1$

e) If Φ *is finite, strictly increasing and continuous, then* H_Φ *is the unique solution of the equation* $\mathbb{E}\left[\Phi\left(X/H_\Phi\right)\right] = 1$

f) If Φ *is convex, then* H_Φ *is convex*

g) If there exists $u_1 < 1$ *with* $\Phi(u_1) < 1$ *and* $u_2 > 1$ *with* $\Phi(u_2) < +\infty$*, then* H_Φ *is convex only if* Φ *is convex.*

If the Orlicz function Φ does not satisfy left-continuity, then properties c) and d) in Proposition 31 may not hold. Notice also that $H_\Phi(X) = 1$ does not imply $\mathbb{E}[\Phi(X)] = 1$ as Example 30.b with $g(x) = x$ and $X \sim U(0, 1)$ shows, since $H_\Phi(X) = 1$ but $\mathbb{E}[\Phi(X)] = 1/2$.

5.2 Characterization results

A classical result about convex Orlicz premia is that they are translation invariant if and only if they 'collapse to the mean', as has been proved in [27] and [25] under the assumption that the Young function Φ is differentiable. Remarkably, enlarging the class of loss functions as in Definition 29 enlarges the class of translation-invariant Orlicz premia as follows.

Theorem 32 *Let Φ and $H_\Phi(X)$ be as in Definition 29, with Φ finite, continuous and strictly increasing. Then:*

a) If H_Φ is translation invariant, then there exist $a > 0$, $b > 0$ and $p \geqslant 0$ such that

$$\Phi(x) = 1 + a(x-1)_+^p - b(x-1)_-^p.$$

b) If H_Φ is translation invariant and convex (resp. concave), then

$$\Phi(x) = 1 + a(x-1)_+ - b(x-1)_-,$$

with $a \geqslant b$ (resp. $b \leqslant a$).

In all cases, letting $q = a/(a+b)$ leads to Examples 26, 27 and 28, so we can conclude that under Definition 29 a translation-invariant Orlicz premium 'collapses' to an L^p-quantile that is necessarily an expectile if additionally convexity or concavity holds.

In the convex case, Orlicz premia are convex and law-invariant return risk measures, so they automatically satisfy the Fatou property and have a dual representation of the form given in Proposition 9, which can be written in a more explicit form as follows.

Proposition 33 *Let Φ and H_Φ be as in Definition 29, with Φ convex. Then,*

$$H_\Phi(X) = \sup_{Q \in \mathbf{P}} \{\tilde{\beta}(Q)\, \mathbb{E}_Q[X]\},$$

where

$$\tilde{\beta}(Q) = \left(\inf_{\lambda > 0} \frac{1}{\lambda} \mathbb{E}\left[\tilde{\Phi}^* \left(\lambda \frac{\mathrm{d}Q}{\mathrm{d}P} \right) \right] \right)^{-1},$$

with $\phi^*(z) := \sup_{x>0}(xz - \phi(x))$ and $\tilde{\Phi}(t) := \Phi(t) - 1$.

We move now to the study of geometrically convex Orlicz premia. It turns out that it is possible to give an equivalent characterization of geometric convexity similar to items f)–g) in Proposition 31, based on the following notion of convexity.

Definition 34 *A function $f\colon [0, +\infty) \to \mathbb{R} \cup \{\pm\infty\}$ is GA-convex if, for each $\lambda \in (0,1)$ and $x, y > 0$, it holds that*

$$f(x^\lambda y^{1-\lambda}) \leqslant \lambda f(x) + (1-\lambda) f(y),$$

where we set by definition $+\infty - \infty = +\infty$.

GA-convexity is a particular type of algebraic convexity owing its name to the presence of both the Geometric and the Arithmetic means in its definition. For similar reasons, geometrically convex functions are sometimes also called GG-convex in the literature. We refer e.g., to [38] and [39] for their properties. Similar to Lemma 4, as a consequence of the AM-GM inequality, a nondecreasing and convex function on $(0, +\infty)$ is GA-convex. For completeness we report the proof in Lemma 40 in the Appendix. The class of increasing GA-convex functions is strictly larger than the class of increasing convex functions, an example of the former being $f(x) = \log x$.

Proposition 35 *Let Φ and $H_\Phi(X)$ be as in Definition 29. If Φ is GA-convex, then H_Φ is geometrically convex. Conversely, if there exists $u_1 < 1$ with $\Phi(u_1) < 1$ and $u_2 > 1$ with $\Phi(u_2) < +\infty$, then H_Φ is geometrically convex only if Φ is GA-convex.*

Summing up, geometrically convex Orlicz premia are a class of return risk measures that is strictly larger than convex Orlicz premia. The latter corresponds to convex Orlicz functions, while the former to a more general situation of GA-convex Orlicz functions.

The multiplicative weighting function in the dual representation of a geometrically convex Orlicz premium can be expressed as

$$\tilde{\alpha}(Q) = \left[\sup_{X|\mathbb{E}[\Phi(X) \leqslant 1]} \exp(\mathbb{E}[\log X]) \right]^{-1}.$$

We end the section by providing an axiomatic foundation for Orlicz risk premia parallel to the axiomatic foundation of generalized shortfall risk measures among law-invariant risk measures given in [19], which is a variant of a well-known result of [45].

Definition 36 *A law-invariant functional ρ has the CxLS property if*

$$\rho(F) = \rho(G) = \gamma \Rightarrow \rho(\lambda F + (1 - \lambda)G) = \gamma,$$

for each $\gamma \in \mathbb{R}$, $F, G \in \mathcal{M}_{1,c}$ and $\lambda \in (0, 1)$.

As is well known, the CxLS property is a necessary condition for elicitability, informally defined as the property of being the minimizer of a suitable expected loss function. We refer e.g., to [24, 5, 47] and the many references therein for the relevance of this notion.

Theorem 37 *Let $\tilde{\rho}\colon L^\infty_{++} \to (0, +\infty)$ be a law-invariant geometrically convex return risk measure with CxLS. Then there exists a GA-convex Orlicz function $\Phi\colon (0, +\infty) \to \mathbb{R} \cup \{+\infty\}$ such that $\tilde{\rho}(X) = H_\Phi(X)$. Furthermore, if $\tilde{\rho}(X) \neq \operatorname{ess\,sup}(X)$, then Φ is finite if and only if $\tilde{\rho}$ has the $\widetilde{WC}$ property.*

Since a convex return risk measure is also geometrically convex, we can derive the following corollary.

Corollary 38 *Let* $\tilde{\rho}\colon L^{\infty}_{++} \to (0,+\infty)$ *be a law-invariant convex return risk measure with CxLS. Then there exists a convex Orlicz function* $\Phi\colon (0,+\infty) \to \mathbb{R} \cup \{+\infty\}$ *such that* $\tilde{\rho}(X) = H_{\Phi}(X)$*.*

As a consequence, we find that generalized Orlicz premia naturally arise as the only elicitable geometrically convex or convex return risk measures. The study of families of loss functions consistent with Orlicz premia and their applications is being pursued in a separate paper.

A Proofs

Proof of Lemma 4. Let $X, Y \in L^\infty_+$ and $\lambda \in (0,1)$. If $\tilde\rho(X) = 0$ or $\tilde\rho(Y) = 0$, the thesis is trivial. By using the AM-GM inequality and the monotonicity and convexity of $\tilde\rho$, it follows that

$$\begin{aligned}\tilde\rho\left(\left(\frac{X}{\tilde\rho(X)}\right)^{\lambda}\left(\frac{Y}{\tilde\rho(Y)}\right)^{1-\lambda}\right) &\leqslant \tilde\rho\left(\lambda\frac{X}{\tilde\rho(X)} + (1-\lambda)\frac{Y}{\tilde\rho(Y)}\right)\\ &\leqslant \lambda\tilde\rho\left(\frac{X}{\tilde\rho(X)}\right) + (1-\lambda)\tilde\rho\left(\frac{Y}{\tilde\rho(Y)}\right) = 1.\end{aligned}$$

Next, from positive homogeneity it follows that

$$\tilde\rho\left(X^{\lambda}Y^{1-\lambda}\right) \leqslant \tilde\rho(X)^{\lambda}\tilde\rho(Y)^{1-\lambda},$$

which completes the proof. ■

Proof of Lemma 7. (i) Let $X_n \in L^\infty_{++}$ satisfy $X_n \xrightarrow{P} X$, $\|X_n\|_\infty \leqslant k$, $X_n \geqslant c > 0$. By the continuous mapping theorem, it follows that $\log(X_n) \xrightarrow{P} \log(X)$, and $\|\log(X_n)\|_\infty \leqslant \max(\log k, -\log c)$, so from the Fatou property of ρ it follows that

$$\rho(\log X) \leqslant \liminf_{n\to+\infty} \rho(\log(X_n)),$$

and exponentiating both sides we get $\tilde\rho(X) \leqslant \liminf_{n\to+\infty} \tilde\rho(\log(X_n))$. The proof of the 'only if' part and the proof of (ii) are similar. ■

Proof of Theorem 8. From Lemmas 5 and 7, it follows that $\rho(X) = \log(\tilde\rho(\exp(X)))$ is a convex monetary risk measure with the Fatou property, so it has the dual representation (see e.g., [23, 18])

$$\rho(X) = \sup_{Q\in\mathbf{P}}\{\mathbb{E}_Q[X] - \alpha(Q)\}, \tag{A.I}$$

where $\alpha(Q) = \sup_{X\in A_\rho} \mathbb{E}_Q[X]$, and A_ρ is the acceptance set of ρ. Since $\tilde\rho(X) = \exp(\rho(\log(X)))$, it follows that

$$\begin{aligned}\tilde\rho(X) &= \exp\left(\sup_{Q\in\mathbf{P}}\{\mathbb{E}_Q[\log X] - \alpha(Q)\}\right) = \sup_{Q\in\mathbf{P}}\left\{\exp\left(\mathbb{E}_Q[\log X] - \alpha(Q)\right)\right\}\\ &= \sup_{Q\in\mathbf{P}}\left\{\tilde\alpha(Q)\exp(\mathbb{E}_Q[\log X])\right\},\end{aligned}$$

where $\tilde\alpha\colon \mathbf{P} \to [0,1]$ is given by

$$\tilde\alpha(Q) = \exp(-\alpha(Q)) = \frac{1}{\exp\left(\sup_{X\in A_\rho}\mathbb{E}_Q[X]\right)} = \frac{1}{\exp\left(\sup_{X\in B_{\tilde\rho}}\mathbb{E}_Q[\log X]\right)}.$$

From $\tilde\rho(1) = 1$, it follows that $\sup_{Q\in\mathbf{P}}\tilde\alpha(Q) = 1$. By Theorem 4.22 and Exercise 4.2.2 in [23], the supremum in (A.I) is attained if ρ has the Lebesgue property. In view of Lemmas 5 and 7, it then follows that the supremum in (4) is attained provided that $\tilde{\rho}$ satisfies the lower-bounded Lebesgue property. ■

Proof of Proposition 9. The first part of the statement is easily derived from Proposition 4.3 in [34]. We can write ([34])

$$\tilde{\beta}(Q) = \sup\{\lambda \geqslant 0 : \lambda\, \mathbb{E}_Q[X] \leqslant \tilde{\rho}(X) \text{ for all } X \in L^\infty_+\}.$$

Therefore, we obtain

$$\begin{aligned}\tilde{\beta}(Q) &= \sup\left\{\lambda \geqslant 0 : \lambda \leqslant \inf_{X\in L^\infty_+} \frac{\tilde{\rho}(X)}{\mathbb{E}_Q[X]}\right\} \\ &= \inf_{X\in L^\infty_+} \frac{\tilde{\rho}(X)}{\mathbb{E}_Q[X]} = \inf_{\tilde{\rho}(X)=1} \frac{1}{\mathbb{E}_Q[X]},\end{aligned}$$

where we use the positive homogeneity of $\tilde{\rho}$ in the last equality. (When $X \equiv 0$, we set $0/0 = 1$.) Now, let us show that

$$\sup_{X\in B_{\tilde{\rho}}} \mathbb{E}_Q[X] = \sup_{\tilde{\rho}(X)=1} \mathbb{E}_Q[X].$$

It directly follows that $\sup_{X\in B_{\tilde{\rho}}} \mathbb{E}_Q[X] \geqslant \sup_{\tilde{\rho}(X)=1} \mathbb{E}_Q[X]$ since $\{X \in L^\infty_+ : \tilde{\rho}(X) = 1\} \subseteq B_{\tilde{\rho}}$. For the other side, let us take $X \in B_{\tilde{\rho}}$ and define $\tilde{X} = \frac{X}{\tilde{\rho}(X)}$. Note that $\tilde{\rho}(\tilde{X}) = 1$. Then, we have

$$\mathbb{E}_Q[X] \leqslant \frac{\mathbb{E}_Q[X]}{\tilde{\rho}(X)} = \mathbb{E}_Q[\tilde{X}] \leqslant \sup_{\tilde{\rho}(Y)=1} \mathbb{E}_Q[Y].$$

Since this inequality is valid for any $X \in B_{\tilde{\rho}}$, if we take the supremum of the left-hand side, we obtain $\sup_{X\in B_{\tilde{\rho}}} \mathbb{E}_Q[X] \leqslant \sup_{\tilde{\rho}(X)=1} \mathbb{E}_Q[X]$, which completes the proof. For the second part, it follows from the proof of Proposition 4.3 in [34] that

$$\tilde{\rho}(X) = \sup_{Z\in H} \mathbb{E}[XZ],$$

where $H = \left\{Z \in L^1_+ : \mathbb{E}[ZY] \leqslant \tilde{\rho}(Y) \text{ for any } Y \in L^\infty_+\right\}$. If we take $Y = 1$, then $\mathbb{E}[Z] \leqslant 1$ for any $Z \in H$, which gives the norm-boundedness of the set H. Furthermore, H is weakly closed, since it is an intersection of weakly closed sets. Let us take a decreasing sequence $(A_n)_n \in \mathcal{F}$ of which the intersection is the empty set. For any $Z \in H$, we have $\mathbb{E}[Z1_{A_n}] \leqslant \tilde{\rho}(1_{A_n})$ for every n. Therefore, by using the Lebesgue property of $\tilde{\rho}$, we have

$$\lim_{n\to+\infty} \sup_{Z\in H} \mathbb{E}[Z1_{A_n}] \leqslant \lim_{n\to+\infty} \tilde{\rho}\left(1_{A_n}\right) = 0,$$

which gives that H is uniformly integrable. Because H is bounded, weakly closed and uniformly integrable, it is weakly compact as a consequence of the Dunford-Pettis theorem (see, e.g., Theorem A.67 in [23]). Therefore, the supremum is attained as a result of the Weierstrass Theorem (see, e.g., Corollary 2.35 in [2]).

Suppose the supremum is attained for $\tilde{Z} \in H$. Then, the supremum is attained for $\tilde{Q}$ such that $\frac{\mathrm{d}\tilde{Q}}{\mathrm{d}P} = \frac{\tilde{Z}}{\mathbb{E}[\tilde{Z}]}$. ■

Proof of Proposition 11. Let us start from the well-known variational formula below linking the exponential certainty equivalent and relative entropy attributed to [20]:

$$\log \mathbb{E}_Q[\exp(Y)] = \sup_{R \ll Q} \{\mathbb{E}_R[Y] - \mathrm{H}(R, Q)\}, \tag{A.II}$$

where $\mathrm{H}(R, Q)$ is the relative entropy or Kullback-Leibler divergence defined in Definition 10.

Letting $X = \exp(Y)$ and exponentiating both sides of (A.II), we obtain

$$\mathbb{E}_Q[X] = \sup_{R \ll Q} \{\tilde{\alpha}(R) \exp(\mathbb{E}_R[\log X])\}, \tag{A.III}$$

where

$$\tilde{\alpha}(R) = \exp(-\mathrm{H}(R, Q)).$$

Now note that $\tilde{\alpha}(R) = 0$ when R is not absolutely continuous with respect to Q. Using this fact, we can rewrite expression (A.III) for $Q \in \mathbf{P}$, as follows:

$$\begin{aligned} \mathbb{E}_Q[X] &= \sup_{R \ll Q} \{\tilde{\alpha}(R) \exp(\mathbb{E}_R[\log(X)])\} \\ &= \sup_{R \in \mathbf{P}} \{\tilde{\alpha}(R) \exp(\mathbb{E}_R[\log(X)])\}, \end{aligned} \tag{A.IV}$$

since we take a supremum of nonnegative numbers, $\tilde{\alpha}(R) = 0$ when $R \notin \mathbf{Q}$ and $\mathbf{Q} \subset \mathbf{P}$, where $\mathbf{Q}$ denotes the set of probability measures absolutely continuous with respect to Q. Substituting the expression for $\mathbb{E}_Q[X]$ derived in (A.IV) in (6), we get

$$\begin{aligned} \tilde{\rho}(X) &= \sup_{Q \in \mathbf{P}} \{\beta(Q) \, \mathbb{E}_Q[X]\} = \sup_{Q \in \mathbf{P}} \left\{\beta(Q) \sup_{R \ll Q} \{\tilde{\alpha}(R) \exp(\mathbb{E}_R[\log X])\}\right\} \\ &= \sup_{Q \in \mathbf{P}} \left\{\beta(Q) \sup_{R \in \mathbf{P}} \{\tilde{\alpha}(R) \exp(\mathbb{E}_R[\log X])\}\right\} \\ &= \sup_{R \in \mathbf{P}} \sup_{Q \in \mathbf{P}} \{\beta(Q) \{\exp(-\mathrm{H}(R, Q)) \exp(\mathbb{E}_R[\log X])\}\} \\ &= \sup_{R \in \mathbf{P}} \{c(R) \exp(\mathbb{E}_R[\log X])\}, \end{aligned}$$

where

$$c(R) = \sup_{Q \in \mathbf{P}} \beta(Q) \exp(-\mathrm{H}(R, Q)).$$

■

Proof of Lemma 13. Take $X, Y \in L^\infty$ such that $X \leqslant Y$. For an arbitrary $x \in \mathbb{R}$, we have $(Y - x)^+ \geqslant (X - x)^+$, which implies $x + \tilde{\rho}((Y - x)^+) \geqslant x + \tilde{\rho}((X - x)^+)$ due to the monotonicity of $\tilde{\rho}$. Since this is valid for any $x \in \mathbb{R}$, by taking the infimum on both sides, we obtain $\rho(X) \leqslant \rho(Y)$. For (b), by using the positive homogeneity of $\tilde{\rho}$ and of the positive part function, we have, for any $\lambda > 0$,

$$\begin{aligned}\rho(\lambda X) &= \inf_{x\in\mathbb{R}} \left\{x + \tilde{\rho}\left((\lambda X - x)^+\right)\right\} = \inf_{x\in\mathbb{R}} \left\{x + \lambda\tilde{\rho}\left(\left(X - \frac{x}{\lambda}\right)^+\right)\right\} \\ &= \inf_{\tilde{x}\in\mathbb{R}} \left\{\lambda\tilde{x} + \lambda\tilde{\rho}\left((X - \tilde{x})^+\right)\right\} = \lambda \inf_{\tilde{x}\in\mathbb{R}} \left\{\tilde{x} + \tilde{\rho}\left((X - \tilde{x})^+\right)\right\} = \lambda\rho(X).\end{aligned}$$

For (c), we have, for any $h \in \mathbb{R}$,

$$\begin{aligned}\rho(X + h) &= \inf_{x\in\mathbb{R}} \left\{x + \tilde{\rho}\left((X + h - x)^+\right)\right\} = \inf_{x\in\mathbb{R}} \left\{x + \tilde{\rho}\left((X - (x - h))^+\right)\right\} \\ &= \inf_{\tilde{x}\in\mathbb{R}} \left\{\tilde{x} + h + \tilde{\rho}\left((X - \tilde{x})^+\right)\right\} = h + \inf_{\tilde{x}\in\mathbb{R}} \left\{\tilde{x} + \tilde{\rho}\left((X - \tilde{x})^+\right)\right\} \\ &= \rho(X) + h.\end{aligned}$$

Finally, let us assume that $\tilde{\rho}$ is convex and take $X, Y \in L^\infty$. Because ρ is positively homogeneous, it is sufficient for (d) to prove that ρ is subadditive. We have

$$\begin{aligned}\rho(X + Y) &= \inf_{x,y\in\mathbb{R}} \left\{x + y + \tilde{\rho}\left((X + Y - x - y)^+\right)\right\} \\ &\leqslant \inf_{x,y\in\mathbb{R}} \left\{x + y + \tilde{\rho}\left((X - x)^+\right) + \tilde{\rho}\left((Y - y)^+\right)\right\} \\ &= \inf_{x\in\mathbb{R}} \left\{x + \tilde{\rho}\left((X - x)^+\right)\right\} + \inf_{y\in\mathbb{R}} \left\{y + \tilde{\rho}\left((Y - y)^+\right)\right\} \\ &= \rho(X) + \rho(Y),\end{aligned}$$

where we have used the convexity and positive homogeneity of $\tilde{\rho}$ and of the positive part function in the second line. ■

Proof of Lemma 15. Note that the functional f is convex since $\tilde{\rho}$ is convex and monotone and the positive part function is convex, and g is convex, too. Then, the inf-convolution of the functionals f and g agrees with the definition of ρ in (7). ■

Proof of Theorem 18. If $\tilde{\rho}$ is law invariant, then, from Lemma 5, the associated convex risk measure ρ given by (2) is also law invariant, and hence has the Kusuoka representation (see e.g., [23, 18])

$$\rho(X) = \sup_{\mu\in\mathcal{M}_1([0,1])} \left(\int_{[0,1]} AV@R_\lambda(X)\mu(\mathrm{d}\lambda) - \beta(\mu)\right),$$

for a suitable $\beta\colon \mathcal{M}_1([0,1]) \to [0, +\infty]$, from which it follows that

$$\begin{aligned}\tilde{\rho}(X) &= \exp\left(\sup_{\mu\in\mathcal{M}_1([0,1])} \left(\int_{[0,1]} AV@R_\lambda(\log X)\mu(\mathrm{d}\lambda) - \beta(\mu)\right)\right) \\ &= \sup_{\mu\in\mathcal{M}_1([0,1])} \tilde{\beta}(\mu) \exp\left(\int_{[0,1]} AV@R_\lambda(\log X)\mu(\mathrm{d}\lambda)\right),\end{aligned}$$

with $\tilde{\beta}(\mu) = \exp(-\beta(\mu))$. From Lemma 7, if $\tilde{\rho}$ has the lower-bounded Lebesgue property, then ρ has the Lebesgue property, and Theorem 35 in [18] implies that $\mu(1) > 0 \Rightarrow \beta(\mu) = +\infty$, from which the thesis follows. ■

Proof of Proposition 19. From Lemma 5 it follows that the corresponding $\rho\colon L^\infty \to \mathbb{R}$ given by equation (2) is a convex law-invariant monetary risk measure. From Proposition 2.1 in [19] suitably adapted to our sign conventions it follows that

$$\lambda \mapsto \rho(\lambda\delta_u + (1-\lambda)\delta_v)$$

is continuous at each $\lambda \in [0, 1)$, for fixed $u, v \in \mathbb{R}$ with $u < v$. Therefore, from the representation (3), we obtain

$$\tilde{\rho}(\lambda\delta_x + (1-\lambda)\delta_y) = \exp(\rho(\lambda\delta_{\log(x)} + (1-\lambda)\delta_{\log(y)})),$$

and the thesis follows by the continuity of compositions with exp and log. ■

Proof of Proposition 22. If ρ is associated to $\tilde{\rho}$ by the correspondence given in (1) and (2), then the $\widetilde{\mathrm{WC}}$ property of $\tilde{\rho}$ is equivalent to the WC property of ρ, similarly defined as the compactness in the $\sigma(L_1, L_\infty)$-topology of the lower level sets of the penalty function in the dual representation of ρ. Indeed,

$$\{Q \in \mathbf{P} | \tilde{\alpha}(Q) \geqslant m\} = \{Q \in \mathbf{P} | \exp(-\alpha(Q)) \geqslant m\} = \{Q \in \mathbf{P} | \alpha(Q) \leqslant -\log(m)\}.$$

So (b) holds if and only if the associated convex risk measure ρ has the WC property. As is well-known (see e.g., [18]), for convex risk measures the WC property is equivalent to the Lebesgue property, so from Lemma 7 it follows that (b) is equivalent to (a). From Proposition 2.7 in [19] adapted to our sign conventions, it follows that the WC property of ρ is equivalent to ψ-weak continuity with respect to some gauge function ψ. From Lemma 4 of [7], it holds that ρ is ψ-weakly continuous if and only if $\tilde{\rho}$ is $\tilde{\psi}$-weakly continuous with $\tilde{\psi}(t) = \psi(\log(t))$, which shows the equivalence between (b) and (c). Furthermore, Proposition 2.7 in [19] shows that the WC property of ρ is equivalent to its mixture continuity for $\lambda \to 1^-$, which combined with Proposition 19 shows that (b) is equivalent to (d). ■

Proof of Proposition 31. (a) From the assumptions on Φ, the set $I_X := \{k > 0 \mid \mathbb{E}[\Phi(X/k)] \leqslant 1\}$ is a nonempty unbounded interval and $\operatorname{ess\,sup} X \in I_X$. Monotonicity of H_Φ follows from the monotonicity of Φ and the proof of positive homogeneity of H_Φ is standard. Since $\mathbb{E}[\Phi(1/k)] \leqslant 1$ for $k \geqslant 1$ and $\mathbb{E}[\Phi(1/k)] > 1$ for $k < 1$, we have $H_\Phi(1) = 1$. (b) From Definition 29, it follows that $X/H_\Phi(X) \leqslant u$, which implies $u \cdot H_\Phi(X) \geqslant \operatorname{ess\,sup} X$, yielding the thesis. (c) Let $g(k) := \mathbb{E}\left[\Phi(X/k)\right]$. Let $k \geqslant H_\Phi(X)$. If $k_n \downarrow k$, then, from left continuity of Φ, it follows that $\Phi(X/k_n) \uparrow \Phi(X/k) \leqslant \Phi(\operatorname{ess\,sup}(X)/k)$. There are two cases now. If Φ does not take the value $+\infty$, from the dominated convergence theorem it follows that g is right-continuous. If $\Phi(x)$ takes the value $+\infty$ for some finite x, then from b) it follows that

$$\Phi\left(\frac{\operatorname{ess\,sup} X}{k}\right) \leqslant \Phi\left(\frac{\operatorname{ess\,sup} X}{\operatorname{ess\,sup} X/u}\right) = \Phi(u) < +\infty,$$

so again the right continuity of g follows from the dominated convergence theorem. Hence, $H_\Phi(X) = \inf\{k \mid g(k) \leqslant 1\} = \min\{k \mid g(k) \leqslant 1\}$. (d) Follows from Definition 29 and (c). (e) Under these assumptions, the function $g(k) = \mathbb{E}[\Phi(X/k)]$ introduced in the proof of (c) is continuous and strictly decreasing, from which the thesis follows. (f) The proof is standard. (g) Assume by contradiction that Φ is not midconvex, i.e., there exist $x_1, x_2 \in (0, +\infty)$ with $\Phi(x_1) < +\infty$, $\Phi(x_2) < +\infty$ such that

$$b := \Phi((x_1 + x_2)/2) > (\Phi(x_1) + \Phi(x_2))/2 := a.$$

We want to prove that there exist $z \in (0, +\infty)$ and $c = \Phi(z)$ such that

$$\begin{cases} \lambda c + (1-\lambda)b > 1 \\ \lambda c + (1-\lambda)a \leqslant 1 \end{cases} \tag{A.V}$$

for some $\lambda \in (0, 1)$, or equivalently that

$$c \in I_\lambda := \left(\frac{1 - b(1-\lambda)}{\lambda}, \frac{1 - a(1-\lambda)}{\lambda} \right].$$

There are three cases. If $a \leqslant 1 < b$, then $c = 1$ satisfies (A.V) for each $\lambda \in (0, 1)$. If $a < b \leqslant 1$, then $z = u_2$ and $c = \Phi(u_2)$ satisfies (A.V) since any $c > 1$ is contained in some I_λ, while if $1 \leqslant a < b$, then $z = u_1$ and $c = \Phi(u_1)$ satisfies (A.V) since any $c < 1$ is contained in some I_λ. As a consequence,

$$\lambda\Phi(z) + (1-\lambda)\Phi((x_1 + x_2)/2) > 1 \geqslant \lambda\Phi(z) + (1-\lambda)\frac{\Phi(x_1) + \Phi(x_2)}{2}. \tag{A.VI}$$

Let $A, B, C \in \mathcal{F}$ be disjoint sets with $P(A) = \lambda$, $P(B) = P(C) = \frac{1-\lambda}{2}$ and let

$$\begin{aligned} X &= z1_A + x_1 1_B + x_2 1_C \\ Y &= z1_A + x_2 1_B + x_1 1_C \\ Z &= z1_A + \frac{x_1 + x_2}{2} 1_{B \cup C} = \frac{X+Y}{2}. \end{aligned}$$

From (A.VI), we have $\mathbb{E}[\Phi(Z)] > 1 \Rightarrow H_\Phi(Z) > 1$ and $\mathbb{E}[\Phi(X)] \leqslant 1 \Rightarrow H_\Phi(X) \leqslant 1$, $\mathbb{E}[\Phi(Y)] \leqslant 1 \Rightarrow H_\Phi(Y) \leqslant 1$, which contradicts the convexity of H_Φ. As a consequence, Φ is midconvex where it is finite, and since it is also nondecreasing it is convex where it is finite. ■

Proof of Theorem 32. Let $x_1 < 1$ and $x_2 > 1$. From the assumptions on Φ, there exists p such that $p\Phi(x_1) + (1-p)\Phi(x_2) = 1$, and from Proposition 31 it follows that $H_\Phi(X) = 1$, where $X = x_1 1_A + x_2 1_{A^c}$ with $P(A) = p$. From positive homogeneity and translation invariance, it follows that $H_\Phi(cX - c + 1) = c - c + 1 = 1$ for any $c \geqslant 0$, so again from Proposition 31 it follows that $p\Phi(c(x_1 - 1) + 1) + (1-p)\Phi(c(x_2 - 1) + 1) = 1$, for any $c \geqslant 0$.
Let $u := x_1 - 1 < 0$ and $v := x_2 - 1 > 0$. The previous arguments have shown

that if

$$pf(u) + (1-p)g(v) = 0, \tag{A.VII}$$

then

$$pf(cu) + (1-p)g(cv) = 0, \text{ for any } c \geqslant 0,$$

where f and g denote the restrictions of the function $h(x) := \Phi(x+1) - 1$ to the domains $(-1, 0)$ and $(0, +\infty)$, respectively. Since from the assumptions on Φ it follows that for any $u \in (-1, 0)$ and $v \in (0, +\infty)$ it is possible to find p such that (A.VII) holds, we find that for each u and v,

$$\frac{f(u)}{g(v)} = \frac{f(cu)}{g(cv)}, \text{ for any } c \geqslant 0,$$

which can be recast into a multiplicative Pexider functional equation (see [1]), whose general solution is

$$\begin{cases} f(u) = -a(-u)^p \text{ if } u \leqslant 0 \\ g(v) = bv^p \text{ if } v \geqslant 0, \end{cases}$$

with $a > 0$, $b > 0$ and $p \geqslant 0$, from which a) follows. To prove b), notice that convexity or concavity holds if and only if $p = 1$, and is determined by the inequality among a and b. ■

Proof of Proposition 33. The penalty function in the general dual representation (6) now takes the following specific form:

$$\frac{1}{\tilde{\beta}(Q)} = \sup_{X \in B_{H_\Phi}} \mathbb{E}_Q[X] = \sup_{X \in L_+^\infty} \{\mathbb{E}_Q[X] : \mathbb{E}[\Phi(X)] \leqslant 1\}.$$

Since Φ is convex, Slater's condition holds and using strong duality, we obtain

$$\begin{aligned} \frac{1}{\tilde{\beta}(Q)} &= \inf_{\lambda > 0} \left\{ \sup_{X \in L_+^\infty} [\mathbb{E}_Q[X] - \lambda\, \mathbb{E}[\Phi(X)] + \lambda] \right\} \\ &= \inf_{\lambda > 0} \left\{ \lambda + \sup_{X \in L_+^\infty} \mathbb{E}\left[\frac{\mathrm{d}Q}{\mathrm{d}P} X - \lambda \Phi(X) \right] \right\} \\ &= \inf_{\lambda > 0} \left\{ \lambda + \lambda \sup_{X \in L_+^\infty} \mathbb{E}\left[\frac{\mathrm{d}Q}{\lambda \mathrm{d}P} X - \Phi(X) \right] \right\} \\ &= \inf_{\lambda > 0} \left\{ \lambda + \lambda\, \mathbb{E}\left[\sup_{x > 0} \frac{\mathrm{d}Q}{\lambda \mathrm{d}P} x - \Phi(x) \right] \right\} = \inf_{\tilde{\lambda} > 0} \frac{1}{\tilde{\lambda}}\, \mathbb{E}\left[\tilde{\Phi}^* \left(\tilde{\lambda} \frac{\mathrm{d}Q}{\mathrm{d}P} \right) \right], \end{aligned}$$

where we used Theorem 14.60 in [43] in the one but last equality, and the change of variable $\lambda = 1/\tilde{\lambda}$ in the last equality. ■

Proof of Proposition 35. We first prove the 'if' part. Let $X, Y \in L^\infty_+$ and $\lambda \in (0,1)$. From the GA-convexity of Φ it follows that

$$\begin{aligned}&\mathbb{E}\left[\Phi\left(\left(\frac{X}{H_\Phi(X)}\right)^\lambda\left(\frac{Y}{H_\Phi(Y)}\right)^{1-\lambda}\right)\right]\\&\leqslant \lambda\,\mathbb{E}\left[\Phi\left(\frac{X}{H_\Phi(X)}\right)\right]+(1-\lambda)\,\mathbb{E}\left[\Phi\left(\frac{Y}{H_\Phi(Y)}\right)\right]\leqslant 1,\end{aligned}$$

which from Proposition 31 implies

$$H_\Phi\left(\frac{X^\lambda Y^{1-\lambda}}{H_\Phi(X)^\lambda H_\Phi(Y)^{1-\lambda}}\right)\leqslant 1,$$

which from positive homogeneity gives

$$H_\Phi\left(X^\lambda Y^{1-\lambda}\right)\leqslant H_\Phi(X)^\lambda H_\Phi(Y)^{1-\lambda}.$$

To prove the 'only if' part, we first assume by contradiction that Φ is not GA-midconvex, i.e., there exist $x_1, x_2 \geqslant 0$ with $\Phi(x_1) < +\infty$ and $\Phi(x_2) < +\infty$ such that $\Phi\left(\sqrt{x_1x_2}\right) > (\Phi(x_1)+\Phi(x_2))/2$. Then, reasoning as in the proof of Proposition 31 item (g), there exist $z \in [0,+\infty)$ and $\lambda \in (0,1)$ such that

$$\lambda\Phi(z)+(1-\lambda)\Phi(\sqrt{x_1x_2}) > 1 > \lambda\Phi(z)+(1-\lambda)\frac{\Phi(x_1)+\Phi(x_2)}{2}. \qquad \text{(A.VIII)}$$

Take disjoint sets $A, B, C \in \mathcal{F}$ with $P(A)=\lambda$, $P(B)=P(C)=\frac{1-\lambda}{2}$ and let

$$\begin{aligned}X&=z1_A+x_11_B+x_21_C\\Y&=z1_A+x_21_B+x_11_C\\Z&=z1_A+\sqrt{x_1x_2}1_{B\cup C}=\sqrt{XY}.\end{aligned}$$

From (A.VIII), we have $\mathbb{E}[\Phi(Z)]>1$ and $\mathbb{E}[\Phi(X)]=\mathbb{E}[\Phi(Y)]<1$, which contradicts with the geometric convexity of H_Φ. As a consequence, Φ is GA-midconvex and since it is also nondecreasing the thesis follows. ■

Proof of Theorem 37. For ease of reference, we recall in the following proposition some of the results given in Theorem 3.10 and Lemma 3.7 of [19], under a different sign convention.

Proposition 39 *Let $\rho\colon L^\infty \to \mathbb{R}$ be a convex law-invariant monetary risk measure with CxLS. Then, there exists a nondecreasing, convex and left-continuous $\varphi\colon \mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ satisfying $\varphi(0)=0$ such that $\rho(X)\leqslant 0 \iff \mathbb{E}[\varphi(X)]\leqslant 0$. Furthermore, if $\rho(X)\neq \operatorname{ess\,sup}(X)$, then φ is finite if and only if ρ has the WC property.*

From the hypotheses and Lemma 5, it follows that ρ given by (2) is a convex law-invariant monetary risk measure, given at the level of distributions by $\rho(F)=\log\left(\tilde{\rho}(F\circ\log)\right)$. If $\rho(F)=\rho(G)=\gamma$, then $\tilde{\rho}(F\circ\log)=\tilde{\rho}(G\circ\log)=\exp(\gamma)$, and since $\tilde{\rho}$ has the CxLS property, $\tilde{\rho}(\lambda(F \circ \log) + (1-\lambda)(G \circ \log)) = \exp(\gamma)$, so $\tilde{\rho}((\lambda F + (1-\lambda)G) \circ \log) = \exp(\gamma)$, implying that also ρ has the CxLS property, hence it satisfies all the assumptions of Proposition 39.

From Proposition 39, it follows that there exists a nondecreasing convex and left-continuous $\varphi \colon \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ satisfying $\varphi(0) = 0$ such that $\rho(X) \leqslant 0$ if and only if $\mathbb{E}[\varphi(X)] \leqslant 0$. From convexity, it follows that φ is continuous where it is finite. Letting $\Phi(x) := 1 + \varphi(\log(x))$, it follows that Φ is an Orlicz function in the sense of Definition 29, and

$$\begin{aligned} \tilde{\rho}(X/k) \leqslant 1 \iff \rho(\log(X/k)) \leqslant 0 \iff \mathbb{E}[\varphi(\log(X/k))] \leqslant 0 \\ \iff \mathbb{E}[\Phi(X/k)] \leqslant 1, \end{aligned}$$

so $\tilde{\rho}(X) = H_\Phi(X)$. Finally, from the convexity of φ, for each $x, y > 0$ and $\lambda \in (0,1)$,

$$\begin{aligned} \Phi(x^\lambda y^{1-\lambda}) &= 1 + \varphi(\log(x^\lambda y^{1-\lambda}) = 1 + \varphi(\lambda \log(x) + (1-\lambda)\log(y)) \\ &\leqslant 1 + \lambda\varphi(\log(x)) + (1-\lambda)\varphi(\log(y)) = \lambda\Phi(x) + (1-\lambda)\Phi(y), \end{aligned}$$

which shows the GA-convexity of Φ. The final part of the thesis follows from Proposition 39 by noticing that $\tilde{\rho}(X) = \operatorname{ess\,sup}(X)$ if and only if $\rho(X) = \operatorname{ess\,sup}(X)$. ■

Proof of Corollary 38. Since a positively homogeneous, monotone convex functional defined on L^∞_{++} is geometrically convex, it follows from Theorem 37 that there exists a nondecreasing GA-convex Orlicz function $\Phi \colon [0, +\infty) \to \mathbb{R} \cup \{+\infty\}$ such that $\tilde{\rho}(X) = H_\Phi(X)$. Since from Proposition 31 H_Φ is convex only if Φ is convex, the thesis follows. ■

Lemma 40 *Let $f \colon (0, +\infty) \to \mathbb{R}$ be nondecreasing and convex. Then, f is GA-convex.*

Proof. For $x, y > 0$ and $\lambda \in (0,1)$ from the AM-GM inequality it holds that $\lambda x + (1-\lambda)y \geqslant x^\lambda y^{1-\lambda}$. Since f is nondecreasing and convex it follows that

$$f(x^\lambda y^{1-\lambda}) \leqslant f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda) f(y),$$

which gives the thesis. ■

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